Some properties for the critical random series-parallel graph

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Outline of the talk

- Recursive tree processes and recursive distributional equations
- Derrida-Retaux recursive system
- Random series-parallel graphes
- Resistances in the critical series-parallel graph
- Distances in the slightly supercritical series-parallel graph

Recursive tree processes and recursive distributional equations

Let $\mathbb{T}$ be an inverted rooted tree. Given $\xi(v)$, $v \in \mathbb{T}$, and $X(u)$, $u \in \mathbb{T}_0$.

Aldous and Bandyopadhyay (2005)introduced

$$X(v) = g(\xi, X(v_1), X(v_2), \cdots), \quad v \in \mathbb{T}/\mathbb{T}_0,$$

and called it the recursive tree process. We consider a max-type function g.

- Height of Galton-Watson tree: $X(v) = 1 + \max\{X(v_i) : \ i \leq \xi(v)\}$
- Range of Branching Random Walk (Biggins 1998):

$$X(v) = \max_i \{(X(v_i) + \xi(v_i))^+\}$$

- Discounted BRW (Athreya 1985): $X(v) = \xi(v) + c \max_i \{X(v_i)\}$.

▶ Recursive distributional equation: $X_{n+1} \overset{d}{=} g(\xi, X_n^{(1)}, X_n^{(2)}, \cdots), \quad n \geq 0.$

Given X_0^*. Let $\xi_0 \sim B(1,p)$ and set $X_0 = \xi_0 X_0^*$. Recursively define

$$X_{n+1} = \max\{X_n^{(1)} + X_n^{(2)} - 1, 0\}, \quad n \geq 0,$$

where $X_n^{(1)}, X_n^{(2)}$ are independent copies of X_n.

- Free energy: $\mathcal{F}_\infty := \lim_{n\to\infty} \downarrow \frac{\mathbf{E}(X_n)}{2^n}$.

Theorem (Collet-Eckmann-Glaser-Martin 1984): $\exists$ Critical value p_c,

$$\mathcal{F}_\infty(p) = 0, \quad p \leq p_c \quad \text{and} \quad \mathcal{F}_\infty(p) > 0, \quad p > p_c.$$

- Indeed, p_c is the solution which satisfies $\mathbf{E}_p(2^{X_0}) - \mathbf{E}_p(X_0 2^{X_0}) = 0$.

Conjecture (Derrida-Retaux 2014) : $\mathcal{F}_\infty(p) = \exp\{-\frac{K+o(1)}{(p-p_c)^{1/2}}\}, \quad p \to p_c^+$.

Theorem 1 (C-Dagard-Derrida-Hu-Lifshits-Shi, 2021) As $p \to p_c^+$,

$$\lim_{n\to\infty} \frac{\mathbf{E}(X_n)}{2^n} = \exp\{-\frac{1}{(p-p_c)^{1/2+o(1)}}\}.$$

Theorem 2 (C-Hu-Shi, 2022): When $p = p_c$,

$$\mathbf{P}(X_n \neq 0) = \frac{1}{n^{2+o(1)}}, \quad \mathbf{E}(X_n) = \frac{1}{n^{2+o(1)}}, \quad n \to \infty.$$

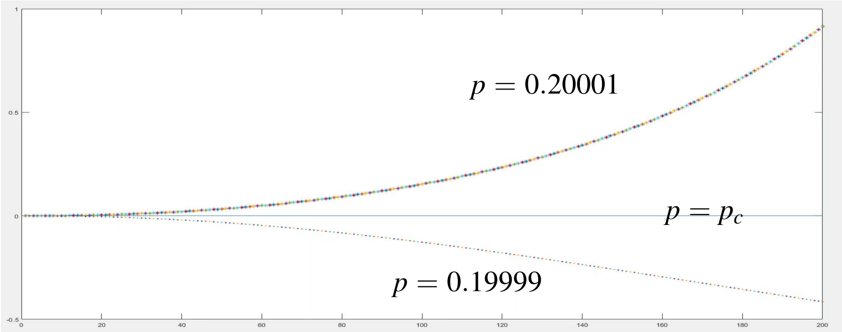
Theorem 3 (C-Hu-Shi, 2023+): As $p \to p_c^-$,

$$-(p-p_c)^{\frac{1}{2}+o(1)} \le \liminf_{n\to\infty} \frac{\log(\mathbf{E}(X_n))}{n} \le \limsup_{n\to\infty} \frac{\log(\mathbf{E}(X_n))}{n} \le -(p-p_c)^{\frac{1}{2}}.$$

Question: Does $\lim_{n\to\infty} \frac{\log(\mathbf{E}(X_n))}{n}$ exists?

Derrida-Retaux recursive system

Figure: $\delta_n := \mathbf{E}_p(X_n 2^{X_n}) - \mathbf{E}_p(2^{X_n})$, where $X_0 = 2\xi_0$ with $\xi_0 \sim B(1,p)$.



If the initial distribution does not exactly lie on the critical manifold, but only in a neighbourhood, with distance $\epsilon := |\delta_0|$, then for a long time, of order $\epsilon^{-1/2}$, the system lies in the ϵ-neighbourhood of the critical manifold before drifting away definitely.

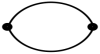
Random series-parallel graph

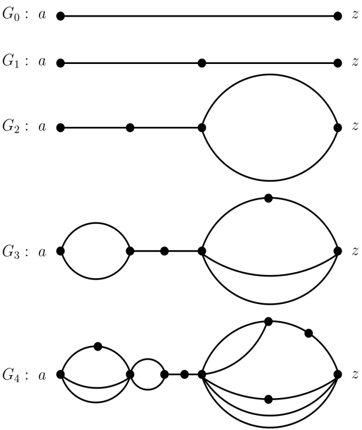
Recursively define a sequence of random series-parallel graphs G_n.

- G_0: a graph with two vertices a and z connected by a single edge;
- To construct G_{n+1} from G_n:

  replace each edge of G_n

  by  with prob. $p \in [0, 1]$,

  or by  with prob. $1 - p$.



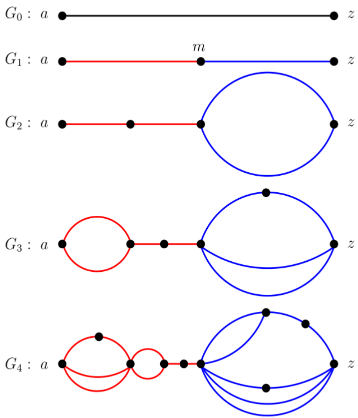
Hambly and Jordan (2004): resistance,

distance, Cheeger constants

Resistances in the critical series-parallel graph

Assume that each edge of G_n has resistance 1, and write R_n for the resistance of a and z in G_n. As Fig, $R_0 = 1,\ R_1 = 2,\ R_2 = \frac{5}{2}$.

$$R_{n+1} \overset{law}{=} \begin{cases} R_n^{(1)} + R_n^{(2)}, & \text{with prob.} \quad p, \\ \frac{1}{\frac{1}{R_n^{(1)}} + \frac{1}{R_n^{(2)}}}, & \text{with prob.} \ 1-p, \end{cases}$$



where $R_n^{(1)}$ and $R_n^{(2)}$ are independent and have the same law as R_n.

- $\mathbf{P}_p(R_n \in \cdot) = \mathbf{P}_{1-p}(\frac{1}{R_n} \in \cdot),\ \forall p$

Resistances in the critical series-parallel graph

Hambly and Jordan (2004): Phase transition for R_n at $p = p_c := \frac{1}{2}$,

$$\lim_n R_n = 0 \text{ a.s. if } p < p_c, \quad \text{and} \quad \lim_n R_n = \infty \text{ a.s. if } p > p_c.$$

- If $p < \frac{1}{2}$ then $\exists \alpha, c \in (0,1), \forall n,\ \mathbf{E}(R_n^\alpha) < c\mathbf{E}(R_{n-1}^\alpha)$
- $\mathbf{P}_p(R_n \in \cdot) = \mathbf{P}_{1-p}(\frac{1}{R_n} \in \cdot), \forall p$

Conjecture (Hambly and Jordan 2004): Under $\mathbf{P}_{p_c}$,

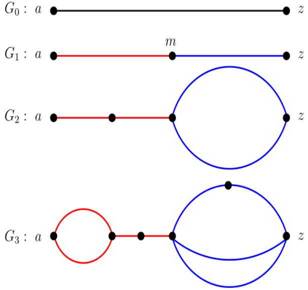
$$R_n \xrightarrow{w} \frac{1}{2}\delta_0 + \frac{1}{2}\delta_\infty.$$

Conjecture (Addario-Berry et al. 2020): $\exists C \in (0, \infty)$, under $\mathbf{P}_{p_c}$

$$\frac{|\log R_n|}{Cn^{1/3}} \xrightarrow{w} \text{a beta distribution.}$$

Distances in the slightly supercritical series-parallel graph

Write D_n for the graph distance of a and z in G_n.



$$D_{n+1} \overset{law}{=} \begin{cases} D_n^{(1)} + D_n^{(2)}, & \text{with prob.} \quad p, \\ \\ \min\{D_n^{(1)}, D_n^{(2)}\}, & \text{with prob.} \;\; 1-p, \end{cases}$$

where $D_n^{(i)}$ are i.i.d $\overset{d}{\sim} D_n$.

Hambly and Jordan (2004): Phase transition at $p_c = \frac{1}{2}$,

$$\lim_n D_n < \infty \;\; \text{a.s.} \quad \text{if } p < p_c, \qquad \text{and} \qquad \lim_n D_n = \infty \;\text{a.s.} \;\; \text{if } p > p_c.$$

- Related to a Galton Watson branching process

Auffinger and Cable (2017), notes by Duquesne and by Jian Ding

Let $\alpha(p) := \lim_n \frac{\log \mathbf{E}(D_n)}{n} \in [0, \infty)$. The limit exists since

$$D_{n+m} \leq \sum_{i=1}^{D_n} \Delta_i,$$

where given D_n, these Δ_i are i.i.d. and have the law of D_m.

Theorem. $\lim_{p \to p_c^+} \frac{\alpha(p)}{\sqrt{p-p_c}} = \frac{\pi}{\sqrt{6}}$.

Heuristics: Let $C := \frac{\pi^2}{6}$. Suppose $f(t,x)$ satisfy $\mathbf{P}(D_n = k) =: \frac{1}{k\sqrt{n}} f(n, \frac{\log k}{\sqrt{n}})$,

$$t\frac{\partial f}{\partial t} = \frac{x}{2}\frac{\partial f}{\partial x} + \frac{f}{2} - Cf\frac{\partial f}{\partial x} + \epsilon t[-2f + 4f\int_0^x f(t,y)dy].$$